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On the Range of Birational Transformation of Curves of Genus Greater than the Canonical Form.

By Virgil Snyder.

In a former paper* I have shown that no curve of order n can be birationally transformed into itself or other curve of order n, if it have fewer than $E\left(\frac{n-1}{2}\right)^2 - 2$ double points, $E(k)$ being the largest integer less than k. In the present paper I show what is the minimum order of the transformed curve, determine the nature of the transformation itself, and show how certain curves of this type can be generated.

1. If a non-singular curve c_n of order n be transformed birationally into c_y by means of adjoint ϕ_x, the minimum value of y is obtained when all the basis points of a net $(\infty^2)\,\phi_x$ are taken upon c_n. This number is $x^2 - x + 1$;† hence

$$y = nx - x^2 + x - 1.$$

Since we exclude collineations, and are concerned with special series g_y^2 only,

$$1 < x \leqq n - 3.$$

Under these conditions y reaches its minimum value $2n - 3$ when $x = 2$. This requires that the net of transforming curves be a system of ∞^2 conics circumscribing a triangle whose vertices lie on c_n; thus the transformation is a quadratic inversion. Hence:

The curve of lowest order into which a non-singular curve of order n *can be transformed by birational transformation other than collineation is of order* $2n-3$, *and the transformation is birational for the entire plane.*

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* Journal, Vol. XXX (1908), pp. 10–18.

† C. Küpper: "Ueber das Vorkommen von linearen Schaaren....," *Sitzungsberichte der Böhmischen Gesellschaft....*, Prag, 1892, pp. 264–272.

But by inversion, the $n-2$ points on each side of the triangle will go into the opposite vertices; hence:

The necessary and sufficient condition that a curve of order $2n-3$ *and genus* $\frac{1}{2}(n-1)(n-2)$ *be birationally transformable into a curve of order* n *is that it have three multiple points of order* $n-2$.

Incidentally, no curve of this nature can also have a linear series g_k^2, k being any integer between n and $2n-3$.

2. This result points out a curious exception to the canonical form of curves of genus p* when $p=6$. The general theorem is that any curve of genus 6 can be reduced to a sextic with four double points, but this is not true of a non-singular quintic, as the simplest curve to which it can be transformed is a curve of order 7, having three triple points. This is the only exception to the general theorem for any genus. *No curve of genus* 6 *can have a linear* g_5^2 *and a linear* g_6^2, *but every such curve has one or the other series.* A c_7 with three triple points is not birationally equivalent to a c_7 of genus 6 with any other configuration of multiple points. Every curve of genus 6 can be transformed to a c_7 without the use of special groups (Clebsch-Lindemann, *l. c.*, p. 689).

3. The same value of y that was determined for $x=2$ is also obtained for $x=n-1$, but this case does not need to be considered, since the special groups can always be defined by simpler curves. However, as an illustration of a net of curves having the maximum number of basis points on a given one, the following curve will be of interest. Consider the curve

$$xy^n + yz^n + zx^n = 0$$

and the linear transformations

$$S\begin{cases} x' = x \\ y' = \theta y \\ z' = \theta^n z \end{cases}, \qquad \theta^{n^2-n+1} = 1, \qquad T\begin{cases} x' = y \\ y' = z \\ z' = x \end{cases}.$$

The curve is invariant under the group generated by S and T. Since $S^{n-1}T = TS$, the group is of order $3(n^2-n+1)$, its operators being of the form S^k, S^lT, S^mT^2. S^k is of order n^2-n+1, the others being of order 3.

The curve is non-singular, $p=\frac{1}{2}n(n-1)$, and the order of S is $2p+1$. The coordinate triangle is invariant under the group. Its sides have n-point contact with the curve at one vertex and a simple intersection at the other.

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* Clebsch-Lindemann: *Vorlesungen über Geometrie*, Vol. I, p. 709. Hyperelliptic curves are excluded.

This accounts for $3(n-2)$ points of inflexion and $\frac{3}{2}(n-2)(n-3)$ double tangents. The remaining $3(n^2-n+1)$ points of inflexion are arranged in three sets of n^2-n+1 each, invariant under S, and also in n^2-n+1 triads invariant under T, one of which is real. From this configuration it follows that if $n>3$, the given curve can not have a larger group than that generated by S and T. If $n=3$ all the inflexions are ordinary; the c_4 is now invariant under the simple group of order 168.

The invariant points of T are $(1,1,1)$, $(1,\omega,\omega^2)$, $(1,\omega^2,\omega)$, $\omega^3=1$. The line joining the imaginary points, $x+y+z=0$, is a bitangent when n is a multiple of 3, the points of contact being the invariant points. The number of bitangents apart from the invariant triangle is $\frac{1}{2}(n^2-n+1)(n^2+3n-10)$. When n is a multiple of 3, this number is not a multiple of 3, but congruent 1, the invariant bitangent under T. The inflexions and bitangents can be curiously arranged on conics of the form

$$(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)-k(x+y+z)^2=0,$$

and n^2-n other systems into which this is transformed by S. But the most important configuration for our purpose is that formed by any point P, and its images under S. We shall first prove the following theorem:

Through the n^2-n+1 points of any cycle of S can be passed ∞^2 curves of order n.

Consider the general equation of a c_n written down with unknown coefficients. It defines $\infty^{\frac{n}{2}(n+3)}$ curves. After passing through n^2-n+1 points, we still require two arbitrary constants, which necessitates that all the determinants of order $\frac{n}{2}(n+3)-1$ in the matrix formed by n^2-n+1 rows and $\frac{n}{2}(n+3)+1$ columns should vanish. This is also a sufficient condition. The successive images of (a,b,c) are $(a, b\theta, c\theta^n),\ldots,(a, b\theta^{n^2-n}, c\theta^{-n})$. When the coordinates of these points are substituted in the equation of c_n it will be seen that the following pairs of coefficients only differ by a constant:

$$x^{n-1}z,\ y^n;\quad x^n,\ yz^{n-1};\quad xy^{n-1},\ z^n;$$

hence not only all the determinants of the matrix vanish, but also all the first and second minors, for at least two columns of each second minor will always be equal. This proves the proposition. Moreover it also follows that not every minor of the third order vanishes.

If this net of c_n be used as transforming curves, the original c_{n+1} is transformed into a curve of order $2n-1$. According to our theorem, the new curve must have three points of multiplicity $n-1$; hence the transforming curves must define three linear series g_n^1, which requires that all the basis points of three pencils must lie on c_{n+1}. These pencils are determined by the n^2-n+1 images of (a, b, c) on c_{n+1} and any vertex of the invariant triangle. For example, the pencil belonging to $(1, 0, 0)$ is

$$a b^{n-1} z^n - c^n x y^{n-1} + k (a^{n-1} c y^n - b^n z x^{n-1}) = 0.$$

The curves of any pencil in the net must have the remaining basis points on a straight line. When the point fixing the pencil is an invariant point, the curves have $(n-1)$-point contact with a side of the triangle. The equations of the transformation may then be written

$$\rho x' = a b^{n-1} z^n - c^n x y^{n-1}, \quad \rho y' = b c^{n-1} x^n - a^n y z^{n-1}, \quad \rho z' = a^{n-1} c y^n - b^n z x^{n-1},$$

from which

$$x' y + y' z + z' x = 0.$$

From these equations, the equation of c_{n+1} and the condition that (a, b, c) is a point upon it, we obtain

$$\sigma x' = a c x z, \quad \sigma y' = a b x y, \quad \sigma z' = b c y z.$$

The original c_{n+1} can be generated by the pencil

$$a b^{n-1} z^n - c^n x y^{n-1} + \lambda (b c^{n-1} x^n - a^n y z^{n-1}) = 0$$

and the projective pencil $cz + \lambda b y = 0$; hence the groups of g_n^1 lie on straight lines passing through the invariant point opposite to the $(n-1)$-point tangent, independently of the point (a, b, c). This completes the reduction of the transformation to the Cremona type.

4. Now suppose c_n has δ distinct double points. In this case

$$y = n x - x^2 + x - 1 - \delta, \qquad \delta < E\left(\frac{n-1}{2}\right)^2,$$

since otherwise y would certainly not be greater than n; this case was considered in my former paper.

If $\delta = 1, 2, 3$, the preceding argument will apply directly; the new curve is of order $2n-4$, $2n-5$, $2n-6$, respectively, and can be obtained by inversion. Since the δ points are assumed to be distinct, the lowest value of y that can be obtained by inversion is $2n-6$. Further, if $\delta \leqq 2(n-4)$, by no other transformation can c_n be reduced to a curve of order as low as $2n-6$, when $n > 8$.

5. For lower values of n, the various cases can be disposed of separately. If $p=5$ and c_n has g_5^2, it must also have a g_3^1 by the Riemann-Roch theorem; hence the standard form of c_6 is one with a triple point. Since $p=2\cdot 5-5$, the lines joining the triads of g_3^1 must all pass through a common point. *A sextic curve with a triple point and two double points can not be birationally transformed into a sextic with any other configuration of multiple points.*

If $p=7$, we can at once say: Any curve of genus 7 can be reduced to c_7 with 8 double points. If these be distinct it can not be further reduced. If c_7 has $2P_3+2P_2$, it also has g_6^2 and can be reduced to a c_6 with 3 double points at the vertices of a triangle. If the $2P_2$ be replaced by a tacnode, c_6 has three collinear double points. If c_7 has P_4+2P_2, c_6 has P_3. These three forms are birationally distinct.

For $p=8$, the canonical series is g_8^2. If g_6^2 exists, g_7^2 must also, but not conversely. If the 13 double points of c_8 are distinct, the c_8 can not have either. Let c_4, c_4' be two quartics intersecting in three points on a given straight line c_1. Through the 13 residual points of intersection, and any four points on c_1 pass a pencil of quintics $c_5+\lambda c_5'$. Make the two pencils projective in such a way that corresponding curves will intersect on c_1. The locus of all the intersections will be a c_9, having c_1 as factor. The resulting c_8 will have at least 13 double points, through which pass a net of quartics, cutting a g_6^2 on c_8, but they can not be used to transform the curve.*

Conversely, a c_6 with two double points can not be birationally transformed into a curve of order 8 with 13 distinct double points. When a binodal c_6 is transformed into c_7, the latter has two triple points.

For $p=9$, g_6^2 and g_7^2 are mutually exclusive. A $c_8^{(9)}$ having g_6^2 must have P_4+2P_3, but a c_7 with 6 double points can be transformed into c_8 with 12 double points by adjoint cubics. The c_8 has the property that a net of adjoint quintics can be passed through the 12 nodes and 9 simple points on the curve. Such a curve can be easily constructed by the above method. When a c_8 of genus 9 is the projection of a space curve of order 9, it can be reduced to a c_7, since when $p=9$, g_7^2 and g_9^3 are reciprocal series by the Riemann-Roch theorem. Conversely, from every c_7 with 6 distinct double points we can define a g_9^3 by means of adjoint ϕ_3; hence when a cubic and a quartic surface intersect in a space cubic curve, the residual intersection is a space curve of order 9, having 19 apparent double

*See Snyder: "On a Special Net of Algebraic Curves," *Bull. Amer. Math. Society*, Vol. XIV (1907), p. 70.

points. Through the 19 bisecants from an arbitrary point can be passed a net of quintic cones.

The larger values of p offer no exception to the general case.

6. If the δ distinct double points be replaced by s_i-fold points such that $\frac{1}{2}\Sigma s_i(s_i-1) \leqq 2(n-4)$, the orders of the transformed curves will be lower than $2n-6$, but, as before, the curve of lowest order can be obtained by inversion, the three multiple points of highest order which are not collinear being the basis points. If $\frac{1}{2}\Sigma s_i(s_i-1) = 2(n-4)$, and ϕ_x has an (s_i-1)-fold point at each s_i-fold point of c_n, then $\frac{1}{2}\Sigma s_i(s_i-1)$ conditions are imposed upon ϕ_x and $\Sigma s_i(s_i-1)$ intersections with c_n are provided for. If now we assume the extreme case of x^2-1 basis points, then

$$y = nx - \Sigma s_i(s_i-1) - \left\{x^2 - 1 - \Sigma \frac{s_i}{2}(s_i-1)\right\} = nx - 2(n-4) - x^2 + 1.$$

When $x = n-3$, $y = n$, but this is only possible when the sum of the three highest s_i is greater than $n-3$, in which case the number of double points would be greater than $2(n-4)$. In every case y is greater than $2n-s_1-s_2-s_3$; hence:

The curve of lowest order into which a curve of order $m > 8$ *and genus* $p \geqq \frac{1}{2}(m-1)(m-2) - 2(m-4)$ *can be birationally transformed can be obtained by quadratic inversion.*

7. It is shown in the theory of space curves that every algebraic space curve can be represented by a cone k_m of the same order as the curve, and the monoid $w = \frac{k_{n+1}}{k_n}$, wherein k_n is a cone containing all the double edges of k_m, and k_{n+1} passes through all the intersections of k_n, k_m. When the curve is given as the complete or partial intersection of two surfaces, the equation of k_m is obtained by eliminating one of the variables (unless the given curve is a conical curve) and the monoid appears incidentally in the process of elimination.*

8. Consider the twisted curve

$$w^{n+1} + a_1 w^n + a_2 w^{n-1} + \ldots = 0, \quad w^2 + b_1 w + b_2 = 0,$$

wherein a_i, b_i are ternary forms of order i. When the intersection is complete, it will be of symbol $(n+1, n+1)$, or say (n, n). If partial, of symbol $(n, n+1)$.

* See Cayley: "On Halphen's Characteristic n....," *Crelle*, Vol. CXI (1893), pp. 347–352.

In the latter case the curve is of order $2n+1$, and has n^2 apparent double points. The n^2 bisecants from any point in space are the basis-edges of a pencil of cones of order n. If the plane projection be c_{2n+1}, and ϕ_n, ϕ'_n be two adjoint curves of order n, then

$$\phi_n \cdot c_1 + c'_1 \cdot \phi'_n = 0$$

will define $\infty^5 \phi_{n+1}$ and in general

$$\phi_n \cdot c_r + \phi'_n \cdot c'_r = 0$$

will define $(r^2 + 3r - 1)$-fold ϕ_{n+r}, wherein

$$1 \overline{\gtrless} r \overline{\gtrless} n - 2.$$

Hence:

*If a curve of order $n + r$ can be passed through $n^2 - \frac{1}{2}(n-r-2)(n-r-1)$ of the n^2 double points of the projection of a space curve of order $2n+1$ and genus $n^2 - n$, then it will pass through all n^2 double points.**

This curve can evidently be birationally transformed into a curve of order $2n$, since the space curve can be projected into such a curve from a point upon it. The question now arises, what is the lowest order to which such a plane curve can be birationally reduced?

The adjoint ϕ_{n+r} has still r^2+3r+1 constants, and the number of variable intersections is $(2n+1)(n+r)-2n^2$. If the order of the transformed curve is y, then all but y of these variable intersections must be fixed basis points, and the system of adjoint ϕ_{n+r} passing through them still have two arbitrary constants.

9. The question may now be stated thus: Given ∞^{r^2+3r+1} adjoint curves of order $n+r$, it is required to find $(2n+1)(n+r) - 2n^2 - y$ fixed basis points upon c_{2n+1}, such that through them will pass ∞^2 curves of the system. This problem is formulated and solved in the Brill-Noether paper in Vol. VII of the *Math. Annalen* (§ 9, p. 290) for the case of a curve of general moduli. Thus, if we put

$$t = r^2 + 3r + 1, \quad R = (2n+1)(n+r) - 2n^2 - y, \quad q = 2,$$

then (D) (p. 291 of the B.-N. paper),

$$R \geqq (q+1)(R - t + q)$$

becomes

$$2y \geqq 4nr + 2n - 7r - 3r^2 + 3.$$

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* R. Sturm: "On Some New Theorems on Curves of Double Curvature," *British Association Report* (1881), p. 146. Sturm's theorem is more general than that here derived, but obtained in a different way. A different proof is given by Noether: "*Zur Grundlegung der Theorie der algebraischen Raumcurven*," *Berliner Abhandlungen*, 1883.

Since we need only consider values of r within the interval $1 \leqq r \leqq n-2$, the minimum value of y is $3n-2$. Hence the method of counting the conditions will be of no service, as it presupposes that c_{2n+1}, $p = n^2 - n$, is a general curve of its class, while our curve is a highly particularized one. For $n=1$ or $n=2$, the basis points may be arbitrary to reduce c_{2n+1} to c_{2n}. For $n=3$, we have c_7 with 9 double points. It is therefore possible to pass a net of ϕ_4 through these 9 double points (which lie on a pencil of cubics) and 4 other points on c_7, thus defining a g_6^2. The transformed c_6 must have a triple point and one double point in order to have a g_7^2.

10. Now let $P \equiv (0, 0, 0, 1)$ be any point not on the space curve R_{2n+1}. Call the cone from this point $k_{2n+1}(x, y, z) = 0$. The simplest monoid will be

$$w = \frac{f_{n+1}(x, y, z)}{f_n(x, y, z)}.$$

f_n passes through the n^2 bisecants from P, and has n other lines in common with k_{2n+1}. f_{n+1} passes through both the n^2 bisecants and the n simple lines common to the other two cones. Let Q be any residual point of intersection of f_{n+1}, k_{2n+1}. The ∞^2 planes through Q will cut R_{2n+1} in a linear g_{2n}^2. Project each of the sections of these planes and the monoid from P, and cut the cones with the plane $w=0$. These plane curves will pass through the n^2 double points of c_{2n+1}, through the n simple points and the projection of the point Q. They are therefore adjoint ϕ_{n+1}, have two degrees of freedom, and have the maximum number of basis points

$$n^2 + n + 1 = (n+1)^2 - (n+1) + 1$$

on c_{2n+1}. When this net is used to transform the curve into c_{2n}, the transforming curves go into the ∞^2 straight lines of the plane, *i. e.*, the space curve is projected from the point Q; hence c_{2n} has one n-fold point, and one $(n-1)$-fold point. Since by partial elimination of w between the equation of the quadric and any $F_{n+1}(x, y, z, w)$ containing R_{2n+1} we can obtain a series of monoids of order $n+r$ and a series of corresponding adjoint curves, ϕ_{n+r}, we say:

The plane projection of the space curve of symbol $(n, n+1)$ on the quadric surface can be birationally transformed into a curve of order $2n$ by means of adjoint curves of order $n+r$, $1 \leqq r \leqq n-2$, having $(n+r)(2n+1) - 2n^2 - 2n$ simple fixed basis points on the given curve. The transformed curve will have one n-fold

point and one $(n-1)$-fold point, and can not be birationally transformed into any simpler curve.

11. If the curve be the complete intersection of a quadric and a surface of order n, $h = n^2 - n$. A cone of order $n-1$ can be passed through the bisecants, and therefore ∞^3 cones of order n. If f_{n-1}, f_n be the lower and upper cones of the simplest monoid passing through the curve, this system of cones may be written

$$f_n + k_1 \cdot f_{n-1} = 0,$$

$k_1 = 0$ being any plane through the common vertex. This is the maximum number of basis points a triply infinite system of curves can have.*

The k_{n-1} passing through the $n^2 - n$ bisecants from P can have no further intersection with k_{2n}. The upper cone f_n will pass through the bisecants, and $2n$ simple edges of the lower cone. The ∞^2 planes through Q will cut the monoid in curves of order n which are projected into adjoint ϕ_n in $w = 0$. They all pass through the image of the point Q; hence, as before, a net of ϕ_n have $n^2 - n + 1$ basis points on c_{2n}. The transformed curve is of order $2n-1$ and is obtained by projecting the twisted curve from Q. Any one of the ∞^3 projecting cones can be taken as superior cone of the minimum monoid.

12. The procedure is now easily generalized. Given any space curve $R_m = 0$ defined by the cone $k_m = 0$ and the monoid

$$w = \frac{f_{n+1}}{f_n}.$$

Since f_n passes through all the bisecants of R_m from P and f_{n+1} through the complete intersection k_m, f_n, and since through this intersection ∞^3 cones of order $n+1$ pass, hence a net of adjoint curves of order $n+1$ always exists which will transform the curve into another, of order $m-1$. For no other curve than those of symbols (n, n), $(n, n+1)$ on the hyperboloid is the maximum number of basis points employed. Let (a, b), $a \leqq b$ be the symbol of a general R_{a+b} on a hyperboloid. The inferior cone of the monoid is f_{b-1}, and the number of basis points for $\infty^2 f_b$ is

$$\frac{a(a-1) + b(b-1)}{2} + a(b-a) + 1.$$

When $a = b$ or $a = b-1$, this number is $b^2 - b + 1$. For other values of a it is smaller.

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* C. Küpper: "Bestimmung der Minimalbasis ....," *Monatshefte der Math. und Physik*, Vol. VI (1895), pp. 5–11.

13. The same reasoning will apply directly to space curves which are the complete intersections of two surfaces, F_{μ}, $F_{\mu'}$. Here

$$m = \mu\mu', \quad n = (\mu - 1)(\mu' - 1), \quad h = \frac{\mu\mu'}{2}(\mu - 1)(\mu' - 1).$$

The lower cone is fixed, and the upper one is fixed by the two conditions of passing through the bisecants, and residual intersection of $k_{\mu\mu'}$, k_n; hence we have but three degrees of freedom, just sufficient to reduce the plane curve to order $\mu\mu' - 1$. The transforming curves are of order $n + 1$. *The plane curves which are the projections of complete intersections of two surfaces of order μ, μ' can not be birationally reduced to order less than $\mu\mu' - 1$.*

As an interesting illustration, consider the two space curves of order 9, the complete intersection of two cubic surfaces, and the quadric curve of type (3, 6). In both cases $h = 18$. In the first, $n = 4$; in the second, $n = 5$. When projected from a point upon it, the first becomes a c_8 with 11 distinct double points; the latter a c_8 with a $P_4 + P_3$. Each can be transformed into another c_8 in an infinite number of ways, but neither can be transformed into any other type, or to a curve of lower order. The first transformation is made by means of ϕ_5, the second by quadratic inversion. The complete intersection of F_{μ}, $F_{\mu'}$ is projected from a point upon it into $c_{\mu\mu'-1}$, having $\frac{mn}{2} + 2 - m$ double points. Through them can always be passed ϕ_n, but not always a net. Thus if $\mu = \mu'$, the minimum curves of transformation are ϕ_{n+1}, and for $\mu' = 2$ they are conics, since the inferior cone of the monoid of a quadric curve, vertex on the curve, is the plane of the two generators through the vertex.

14. In general, the ∞^1 transformations of the projection of any R_m into c_{m-1} may be effected as follows: Let Q, S be any two points on the curve. By means of the sections of the monoid from any point P by the ∞^2 planes through Q we have already one such transformation. Similarly for the sections of the same monoid by the net of planes through S; hence these sections will define g^2_{m-1} on the c_{m-1} from P; but this can be more simply done by the sections of the monoid from P. The planes through SP will define a pencil of lines through the image of S. The others will project into ϕ_{x+1}, ϕ_x being the minimum cone through the trisecants passing through P.

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